Chapter 3: Complex Analysis

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Complex functions

Consider a set $D \subseteq \mathbb{C}$.

$$f : D \to \mathbb{C},$$

where $f(z) = w$, with $z \in D$ and $w \in \mathbb{C}$.
Since we can write $z = x + iy$, then

$$w = u + iv = u(x, y) + iv(x, y),$$

hence,

$$f(z) = f(x + iy) = u(x, y) + iv(x, y).$$

In that case, $u(x, y)$ and $v(x, y)$ are real functions.

Examples

Example 1. Consider $f(z) = z^4$. Take $z = x + iy$. Then,

Examples

Example 2. Consider $f(z) = \overline{z}\text{Re}(z) + z^2 + \text{Im}(z)$. Then, In the last two examples we write $f(z)$ like $f(x+iy)$. Now, if we want to write $f(x+iy)$ in terms of z, we need to write

$$x = \frac{z+\overline{z}}{2}, \quad y = \frac{z-\overline{z}}{2i}.$$

Examples

Example 3. Consider $f(z) = 4x^2 + i4y^2$. Then,

Limits and Continuity

Let $f(z) = u(x,y) + iv(x,y)$ be a complex function that is defined in some neighborhood of $z_0 = x_0 + iy_0$, except perhaps in z_0. Then,

$$\lim_{z \to z_0} f(z) = w_0 = u_0 + iv_0$$

if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0, \text{ and } \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0.$$

Example. Consider $f(z) = z^2 - 2z + 1$ and calculate the limit when $z \to 1 + i$.

Since

$$f(z) = z^2 - 2z + 1 = x^2 - y^2 - 2x + 1 + i(2xy - 2y),$$

$$\lim_{(x,y)\to(1,1)} u(x,y) = 1 - 1 - 2 + 1 = -1,$$

$$\lim_{(x,y)\to(1,1)} v(x,y) = 0.$$

Then, $\lim_{z\to 1+i} f(z) = -1$.

Example. Calculate $\lim_{z\to i} \frac{z^4-1}{z-i}$. Then,

Example. Calculate $\lim_{z\to 1+i} \frac{z^2+z-2+i}{z^2-2z+2}$. Then,

Properties about the limits and continuity

Result 1. Let $\lim_{z\to z_0} f(z) = a$ and $\lim_{z\to z_0} g(z) = b$. Then,

- $\lim_{z\to z_0}(f(z) \pm g(z)) = a \pm b$.
- $\lim_{z\to z_0}(f(z)g(z)) = ab$.
- $\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{a}{b}$.

Result 2. Let $f(z)$ be a complex function that is defined for all values of z in some neighborhood of z_0. We say that f is continuous at z_0 if the following are satisfied:

- $\lim_{z\to z_0} f(z)$ exists.
- $f(z_0)$ exists.
- $\lim_{z\to z_0} f(z) = f(z_0)$.

Properties about the limits and continuity

Result 3. Let $f(z) = u(x, y) + iv(x, y)$ be defined in some neighborhood of z_0. Then, f is continuous at $z_0 = x_0 + iy_0$ if and only if u and v are continuous at (x_0, y_0).

Example. Study the continuity of $f(z) = \frac{z^2-2i}{z^2-2z+2}$ at $z_0 = 1 + i$.